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1984 J. Phys. A: Math. Gen. 17 L825

(http://iopscience.iop.org/0305-4470/17/15/002)

J. Phys. A: Math. Gen. 17 (1984) L825–L829. Printed in Great Britain

LETTER TO THE EDITOR

Nonlinear Schrödinger equation, Painlevé test, Bäcklund transformation and solutions

W-H Steeb, M Kloke, and B-M Spieker

Universität Paderborn, Theoretische Physik, D4790 Paderborn, West Germany

Received 2 August 1984

Abstract. We demonstrate that the nonlinear Schrödinger equation passes the Painlevé test and construct a Bäcklund transformation and solutions within this approach. The connection with the Hirota bilinear formalism is discussed. Also the Heisenberg ferromagnet equation which is gauge equivalent to the nonlinear Schrödinger equation is studied.

It is well known that the nonlinear Schrödinger equation in one space dimension

$$\mathrm{i}w_t + w_{xx} + aw(w^*w) = 0, \qquad (a \in \mathbb{R}) \tag{1}$$

can be solved with the help of the inverse scattering transform and that there is an auto Bäcklund transformation. For $a > 0$, equation (1) has an N-envelope soliton solution. For $a < 0$, there is a dark pulse solution (envelope–hole solution).

In the present letter we demonstrate that this equation passes the Painlevé test. We derive a Bäcklund transformation and describe the connection with the Hirota bilinear formalism. The nonlinear Schrödinger equation in one space dimension and the Heisenberg ferromagnet equation in one space dimension are gauge equivalent. Hence, we also perform a Painlevé test for this equation.

Stimulated by results obtained from the inverse scattering theory for soliton equations, a so-called Painlevé test for the integrability of a given dynamical system was made, i.e. every group theoretical reduction of a completely integrable field equation to an ordinary differential equation (ODE) should have the Painlevé property (Ablowitz and Segur 1981). This means that the only movable singularities of all its solutions are poles. This property can be proved in a straightforward manner (so-called singular point analysis) by expanding the solutions into Laurent series and investigating whether these expansions contain enough arbitrary constants to cover the entire solution manifold of the equation. A review about soliton equations, group theoretical reductions and the Painlevé test has been given by Lakshmanan and Kaliappan (1981). The conjecture given above can serve (if it is true) to test whether a partial differential equation (PDE) or a system of PDE's is non-integrable. If we find that at least one of the systems of ODE's (resulting from group theoretical reductions) does not have the Painlevé property, then we can conclude that the system of PDE's is not completely integrable. On the other hand, if we find that all systems of ODE's have the Painlevé property, then we cannot conclude in general that the system of PDE's is completely integrable.

Recently, Ward (1984) has introduced the Painlevé property for PDE's. The system of PDE's under investigation is considered in the complex domain. Let n be the number of the independent variables. Assume that the system of PDE's has coefficients which are analytic on C^n. The Painlevé property is defined as follows: if S is an analytic non-characteristic complex hypersurface in C^n, then every solution of the PDE which is analytic on $C^n \backslash S$, is meromorphic on C^n.

A weaker form of the Painlevé property was proposed by Weiss *et al* (1983). They looked for solutions of the PDE in the form

$$u = \Phi^n \sum_{j=0}^{\infty} u_j \Phi^j \tag{2}$$

where Φ is the analytic function whose vanishing defines a non-characteristic hypersurface S. Inserting this expansion into the PDE leads to conditions on n and recursion relations for the functions u_j. The Painlevé property here states that n should be an integer, that the recursion relations should be consistent and that the series expansion (2) should contain the correct number of arbitrary functions. Resonances are those values of j at which it is possible to introduce arbitrary functions into the expansion (2). Notice that it may happen that more than one branch arises. The expansion (2) could, *a priori*, miss essential singularities. This behaviour is well known for ODE's. If we study the case with more than one field, then the expansion is given by

$$u_k = \Phi^{n_k} \sum_{j=0}^{\infty} u_{kj} \Phi^j. \tag{3}$$

Bäcklund transformations for the PDE which has the Painlevé property can be found when a suitable cut-off is possible for the series (2) (or series (3)). Meanwhile, various authors (compare Oevel and Steeb (1984) and references therein) have applied this approach.

The motivation of the ansatz (2) (and analogously for ansatz (3)) comes from the theory of ODE's where a necessary condition for

$$\mathrm{d}^n w/\mathrm{d}z^n = F(z, w, \ldots, \mathrm{d}^{n-1}w/\mathrm{d}z^{n-1}) \tag{4}$$

(F rational in $w, \ldots, \mathrm{d}^{n-1}w/\mathrm{d}z^{n-1}$, and analytic in z) to have the Painlevé property is that there is a Laurent expansion with $(n-1)$ arbitrary expansion coefficients. Laurent series must be obtained near every possible movable singularity type of the ODE. It may happen that more than one branch arises. A necessary condition for the existence of a sufficient number of algebraic first integrals is given for a class of system of ODE's by Yoshida (1983a, b). He proved that in order that a given system of ODE's is algebraically integrable, all possible resonances (Yoshida calls them Kowalevski exponents) must be rational numbers.

It is conjectured that if a field equation has the Painlevé property then this equation is integrable. On the other hand we cannot conclude in general that a PDE which is integrable has the Painlevé property. An example is the Harry Dym equation $u_t = u^3 u_{xxx}$ (Weiss 1983). Another example is the nonlinear diffusion equation $u_t = (u^{-2}u_x)_x$ (Steeb and Strampp 1984).

Now let us perform our Painlevé test for the nonlinear Schrödinger equation. We put $w = u + \mathrm{i}v$, where u and v are real fields. Then we obtain

$$\begin{aligned} &u_t + v_{xx} + av(u^2 + v^2) = 0 \\ &v_t - u_{xx} - au(u^2 + v^2) = 0. \end{aligned} \tag{5}$$

The Painlevé test for PDE's can be performed in the same manner as the singular point analysis for ODE's. First we determine the dominant behaviour. Inserting $u \sim \Phi^n u_0$, $v \sim \Phi^m v_0$ into equation (5) (considered in the complex domain) we obtain $n = m = -1$ and the 'expansion coefficients' u_0 and v_0 are determined by $(j=0)$

$$2\Phi_x^2 = -a(u_0^2 + v_0^2). \tag{6}$$

This result indicates that u_0 or v_0 can be chosen arbitrarily and therefore $r=0$ is a resonance. Determining the resonances we obtain $r_1 = -1$, $r_2 = 0$, $r_3 = 3$, and $r_4 = 4$. Inserting ansatz (3) into equation (5) we find at $j = 1$ that

$$\begin{pmatrix} 2\Phi_x u_{0x} + \Phi_{xx} u_0 - \Phi_t v_0 \\ 2\Phi_x v_{0x} + \Phi_{xx} v_0 + \Phi_t u_0 \end{pmatrix} = \begin{pmatrix} 2(au_0^2 - \Phi_x^2) & 2au_0 v_0 \\ 2au_0 v_0 & 2(av_0^2 - \Phi_x^2) \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}. \tag{7}$$

At $j = 2$ we find

$$\begin{pmatrix} u_{0xx} - v_{0t} + 2au_1(u_1 u_0 + v_1 v_0) + au_0(u_1^2 + v_1^2) \\ v_{0xx} + u_{0t} + 2av_1(u_1 u_0 + v_1 v_0) + av_0(u_1^2 + v_1^2) \end{pmatrix}$$
$$= \begin{pmatrix} 2(au_0^2 - \Phi_x^2) & 2au_0 v_0 \\ 2au_0 v_0 & 2(av_0^2 - \Phi_x^2) \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}. \tag{8}$$

At the resonance $r_3 = 3$ we obtain

$$\begin{pmatrix} -u_{1xx} + v_{1t} - au_1(u_1^2 + v_1^2) + \Phi_t v_2 - \Phi_{xx} u_2 - 2\Phi_x u_{2x} - 2au_2(3u_0 u_1 + v_0 v_1) \\ \qquad - 2av_2(u_0 v_1 + v_0 u_1) \\ -v_{1xx} - u_{1t} - av_1(u_1^2 + v_1^2) - \Phi_t u_2 - \Phi_{xx} v_2 - 2\Phi_x v_{2x} - 2av_2(3v_0 v_1 + u_0 u_1) \\ \qquad - 2au_2(u_0 v_1 + v_0 u_1) \end{pmatrix}$$
$$= \begin{pmatrix} 2au_0^2 & 2au_0 v_0 \\ 2au_0 v_0 & 2av_0^2 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}. \tag{9}$$

Obviously, the rank of the matrix on the right-hand side is equal to one, as it must be. Inserting u_0, v_0, u_1, v_1, u_2, v_2 into equation (9) we find that both equations coincide; thus u_3 or v_3 can be chosen arbitrarily. At the resonance $r_4 = 4$ we find the same result. When we insert $u_0, \ldots, v_3$ we find that u_4 or v_4 can be chosen arbitrarily. We find that equation (5) has the Painlevé property in the weaker sense. We notice that the nonlinear Schrödinger equation in two and three space dimensions does not pass the Painlevé test.

A Bäcklund transformation can be found as follows: from the above result (equation (9)) we see that u_1 and v_1 satisfy the nonlinear Schrödinger equation when we put $u_j = 0$ and $v_j = 0$ for $j \geq 2$. Consequently, we insert

$$u = \Phi^{-1} u_0 + u_1, \qquad v = \Phi^{-1} v_0 + v_1 \tag{10}$$

into equation (5). It follows that

$$\begin{aligned} &[-2\Phi\Phi_{xx} + 2\Phi_x^2 + a(u_0^2 + v_0^2)]v_0 + (-\Phi_t u_0 + \Phi u_{0t} + \Phi v_{0xx} + \Phi_{xx} v_0 - 2\Phi_x v_{0x})\Phi \\ &\qquad + [2av_0(u_0 u_1 + v_0 v_1) + av_1(u_0^2 + v_0^2)]\Phi + 2av_1(u_0 u_1 + v_0 v_1)\Phi^2 \\ &\qquad + [u_{1t} + v_{1xx} + av_1(u_1^2 + v_1^2)]\Phi^3 = 0 \\ &[2\Phi\Phi_{xx} - 2\Phi_x^2 + a(u_0^2 + v_0^2)]u_0 + (-\Phi_t v_0 + \Phi v_{0t} - \Phi u_{0xx} - \Phi_{xx} u_0 + 2\Phi_x u_{0x})\Phi \\ &\qquad + [-2au_0(u_0 u_1 + v_0 v_1) - au_1(u_0^2 + v_0^2)]\Phi - 2au_1(u_0 u_1 + v_0 v_1)\Phi^2 \\ &\qquad + [v_{1t} - u_{1xx} - au_1(u_1^2 + v_1^2)]\Phi^3 = 0. \end{aligned} \tag{11}$$

Let us assume that u_1 and v_1 satisfy the nonlinear Schrödinger equation (4). We choose $u_1 = v_1 = 0$. Then equation (11) takes the form

$$\begin{aligned}&[-2\Phi\Phi_{xx}+2\Phi_x^2+a(u_0^2+v_0^2)]v_0+(-\Phi_t u_0+\Phi u_{0t}+\Phi v_{0xx}+\Phi_{xx}v_0-2\Phi_x v_{0x})\Phi=0\\&[2\Phi\Phi_{xx}-2\Phi_x^2+a(u_0^2+v_0^2)]u_0+(-\Phi_t v_0+\Phi v_{0t}-\Phi u_{0xx}-\Phi_{xx}u_0+2\Phi_x u_{0x})\Phi=0.\end{aligned} \tag{12}$$

When we introduce the Hirota bilinear operators (Hirota 1974) according to

$$D_t^n D_x^m f\circ g := (\partial/\partial t-\partial/\partial t')^n(\partial/\partial x-\partial/\partial x')^m(f(x,t)g(x,t))_{x'=x,t'=t}, \tag{13}$$

equation (12) can be written as

$$\begin{aligned}&[-D_x^2\Phi\circ\Phi+a(u_0^2+v_0^2)]v_0+(D_t u_0\circ\Phi+D_x^2 v_0\circ\Phi)\Phi=0\\&[D_x^2\Phi\circ\Phi-a(u_0^2+v_0^2)]u_0+(D_t v_0\circ\Phi-D_x^2 u_0\circ\Phi)\Phi=0.\end{aligned} \tag{14}$$

These equations can be decoupled according to

$$\begin{aligned}&(D_x^2\Phi\circ\Phi-\mu)=a(u_0^2+v_0^2)\\&(D_t u_0\circ\Phi+D_x^2 v_0\circ\Phi)=0\\&(-D_t v_0\circ\Phi+D_x^2 u_0\circ\Phi-\mu)=0\end{aligned} \tag{15}$$

where μ is a real constant to be determined. In particular we have

$$(u^2+v^2)=\mu/2-(\ln\Phi)_{xx}. \tag{16}$$

To construct solutions we expand Φ, u_0 and v_0 as power series, as described by Hirota (1974). Inserting these power series into equation (15) we can easily construct N-soliton solutions.

A comment is in order about the Hirota bilinear formalism. In this approach Hirota (1974) introduces new dependent variables as described for the nonlinear Schrödinger equation. This ansatz is always motivated by the Painlevé test. When we consider, for example, the Kadomtsev–Petviashvili equation

$$u_{tx}+uu_{xx}+(u_x)^2+u_{xxxx}+u_{yy}=0 \tag{17}$$

the Painlevé test tells us that $n=-2$, $u_0=-12\Phi_x^2$ and $u_1=12\Phi_{xx}$. The cut-off of the series (2) with $u_j=0$ for $j\geq 3$ leads to

$$u=\Phi^{-2}u_0+\Phi^{-1}u_1+u_2 \tag{18}$$

where u_2 satisfies equation (17). Setting $u_2=0$ we obtain $u=12(\ln\Phi)_{xx}$. Then Hirota (1974) writes down the bilinear equation with the help of Φ.

Now we want to apply the Painlevé test to the classical Heisenberg ferromagnet equation in one space dimension. This equation is given by

$$S_t=S\times S_{xx} \tag{19}$$

where $S=(S_1,S_2,S_3)$, $S_1^2+S_2^2+S_3^2=1$ and x denotes the vector product. We know that this equation is completely integrable. The natural boundary conditions are $S(x,t)\to(0,0,1)$ as $|x|\to\infty$. We cannot apply the Painlevé test directly to equation (19) since the condition $S^2=1$ is not included. Thus we perform a stereographic projection

$$S_1=2u/Q,\qquad S_2=2v/Q,\qquad S_3=(-1+u^2+v^2)/Q, \tag{20}$$

where $Q=1+u^2+v^2$. The we obtain

$$\begin{aligned} &Qv_t+Qu_{xx}-2(u_x^2-v_x^2)u-4vu_xv_x=0 \\ &-Qu_t+Qv_{xx}+2(u_x^2-v_x^2)v-4uu_xv_x=0. \end{aligned} \tag{21}$$

The dominant behaviour is given by $n=m=-1$. The resonances are $r_1=-1$ (double), $r_2=0$ (double). At $r_2=0$ we find that u_0 and v_0 can be chosen arbitrarily. Summing up, we find a solution of the form (3) with three arbitrary functions, namely u_0, v_0 and Φ. Nevertheless, we say that equation (21) passes the Painlevé test.

The Heisenberg ferromagnet equation and the nonlinear Schrödinger equation are gauge equivalent (Zakharov and Takhtadzhyan 1979). Both equations arise as consistency conditions of a system of linear PDE's

$$\psi_x=U\psi, \qquad \psi_t=V\psi \tag{22}$$

where $\psi=(\psi_1,\ldots,\psi_n)$ $(n=2)$ and U and V are 2×2 matrices. The consistency condition is given by

$$U_t-V_x+[U,V]=0. \tag{23}$$

Two systems of nonlinear PDE's that are integrable by the inverse scattering transform are said to be gauge equivalent if there is an $n\times n$ matrix g (invertible) which depends on x and t such that

$$U_1=gU_2g^{-1}+g_xg^{-1}, \qquad V_1=gV_2g^{-1}+g_tg^{-1}. \tag{24}$$

Consequently, for two gauge equivalent soliton equations the resonances are not the same in general.

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